

# Algorithm for computing Semi-Fourier sequences of Expressions involving exponentiations and integrations

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## Abstract

We provide an algorithm for computing semi-Fourier sequences for expressions constructed from arithmetic operations, exponentiations and integrations. The semi-Fourier sequence is a relaxed version of Fourier sequence for polynomials (expressions made of additions and multiplications).

## 1 Introduction

The main contribution of this paper is an algorithm for computing semi-Fourier sequences for expressions constructed from arithmetic operations, exponentiations and integrations. The semi-Fourier sequence is a relaxed version of Fourier sequence for polynomials (expressions made of additions and multiplications). Below we elaborate on the above statements.

Fourier sequence of polynomials is a finite sequence of successive differentiation. As an example, consider

$$e = x^3 + 3x^2 + 5x + 7$$

Then the Fourier sequence of  $e$  is given by

$$\begin{aligned} g_1 &= e &= x^3 + 3x^2 + 5x + 7 \\ g_2 &= g'_1 &= 3x^2 + 6x + 5 \\ g_3 &= g'_2 &= 6x + 6 \\ g_4 &= g'_3 &= 6 \end{aligned}$$

It has an obvious but important property: the successive differentiation eventually leads to a constant. It also has another nice property due to Budan-Fourier [1, 2]: Let  $\nu(x)$  denote the number of sign changes in the sequence  $g_1(x), \dots$  (with zeros ignored). Then the number of roots of  $e$  in  $(a, b]$ , counted with multiplicities, is equal to  $\nu(a) - \nu(b) - 2s$ , where  $s$  is a nonnegative integer.

One naturally wonders whether one can do the same for non-polynomials. Consider

$$e = \exp(x) + \exp(x^2)$$

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Let us compute the successive differentiation as we did above. We obtain

$$\begin{aligned} g_1 &= e = \exp(x) + \exp(x^2) \\ g_2 &= g'_1 = \exp(x) + 2x \exp(x^2) \\ g_3 &= g'_2 = \exp(x) + 2 \exp(x^2) + 4x^2 \exp(x^2) \\ &\vdots \end{aligned}$$

Note that the successive differentiation will *not* lead to a constant. Thus Fourier sequence for  $\exp(x) - \exp(x^2)$  does *not* exist.

In [5, 6], Strzeboński (one of the co-authors of this paper) relaxed the notion of Fourier sequence to “semi”-Fourier sequence. For the above example, a semi-Fourier sequence is given by

$$\begin{aligned} g_1 &= e \cdot h_1 = 1 + \exp(x^2) \exp(x)^{-1} & h_1 &= \exp(x)^{-1} \\ g_2 &= g'_1 \cdot h_2 = 2x - 1 & h_2 &= \exp(x^2)^{-1} \exp(x) \\ g_3 &= g'_2 \cdot h_3 = 2 & h_3 &= 1 \end{aligned}$$

Note that before/after each differentiation one is allowed to multiply by another expression,  $h_i$ , which is non-zero for all values of  $x$  in the domain of  $e$ . Of course when we require that  $h_i = 1$ , then it is a Fourier sequence. By allowing  $h_i$  to be other than 1, one obtains a relaxed version of Fourier sequence. Since  $h_i$  is non-zero for all real values of  $x$ , the semi-Fourier sequence still preserves the Budan-Fourier’s property mentioned above (Theorem 2.4 in [5]).

Strzeboński also proved constructively that a semi-Fourier sequence exists for expressions constructed from arithmetic operations, exponentiations, logarithms and arctangents, by providing an algorithm for computing such a sequence.<sup>1</sup>

One again naturally wonders whether semi-Fourier sequences exist for even larger class of functions and whether one can find them algorithmically. One natural class to consider is the set (called Exp-Int) of expressions constructed from arithmetic operations, exponentiations and integrations (int). It is easy to see that logarithms and arctangents belong to Exp-Int, since they are integrals of  $1/x$  and  $1/(1+x^2)$ . However there are many expressions in Exp-Int that *cannot* be expressed in terms of arithmetic operations, exponentiation, logarithms and arctangents, such as

$$\begin{aligned} e &= \exp(x \operatorname{int}(\exp(-x^2))) - \operatorname{int}(\exp(-x^2)) - 3 \\ &= \exp\left(x \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)\right) - \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) - 3 \end{aligned}$$

Many elementary and special functions can be represented as Exp-Int expressions, such as rational functions, exponential functions, logarithm, radicals ( $\sqrt[n]{\phantom{x}}$ ), inverse trigonometric functions (arcsin, arccos, arctan), (inverse) hyperbolic trigonometric functions (sinh, cosh, tanh, arsinh, arcosh, artanh), hyperbolic sine/cosine integrals (Shi, Chi), logistic sigmoid function ( $F$ ), error function (erf), logarithmic integral function (li), exponential integral function (Ei), polylogarithm function ( $\operatorname{Li}_s$ ), Spence function ( $\operatorname{Li}_2$ ), Gudermannian function (gd), Dawson function ( $D_{\pm}$ ), Kummer function ( $\Lambda_n$ ), Incomplete beta function ( $B_{ab}$ ), Incomplete gamma function ( $\Gamma_s$ ), Incomplete elliptic integral ( $F_k$ ), etc.

The main contribution of this paper is to prove constructively that there exists a semi-Fourier sequence for every Exp-Int expression, by providing an algorithm for computing such a sequence. For a semi-Fourier sequence computed by the algorithm for the above  $e$ , see Example 2.

Of course, the main task of such an algorithm is figuring out  $h_i$ ’s at each step, so that the sequence eventually terminates. It was quite challenging because most choices of  $h_i$ ’s lead to infinite sequences and

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<sup>1</sup>The definition of a semi-Fourier sequence in this paper is a bit stricter than the definition used in [5, 6], however the semi-Fourier sequences constructed in [5, 6] satisfy the stricter requirements.

because, at each step, it was by no means obvious to predict which  $h_i$ 's would be the lucky ones. We overcame the challenge by defining a partial ordering (ranking) on expressions, and ensuring that the rank decreases.

One again wonders whether there is even larger class of functions for which semi-Fourier sequences exist. It is easy to see that semi-Fourier sequences exist for any real analytic function  $e$  with finitely many roots, since one can take  $g_1$  to be a polynomial which has the same roots as  $e$  with the same multiplicities. However, in general  $g_1$  cannot be constructed algorithmically, since we cannot compute the roots of  $e$  (in fact, the original reason for introducing semi-Fourier sequences was to be able to isolate roots algorithmically). Finding a larger class of functions for which semi-Fourier sequences exist and can be found algorithmically is an open problem which we leave for future work.

There are several closely related works. As mentioned above, in [5, 6], Strzeboński, one of the authors of the present paper, introduced the notions, and algorithms for semi-Fourier sequence for exponential - logarithmic - arctan expressions and showed that the sequences have the Budan-Fourier property. This present paper gives an algorithm for larger class of expressions. By construction, it is immediate that the output sequences have the Budan-Fourier property. In [5, 6], Strzeboński also gave real root isolation algorithms. However, in the present paper, we do not do so, mainly because such real root isolation algorithm would require zero-testing which is non-trivial for Exp-Int and interesting on its own, deserving a separate paper. Hence we leave it for the future research.

In [4], Richardson gave an algorithm for recognizing zeros among exponential-logarithmic expressions. For this, he generates certain sequences of expressions. They are similar but not the same as a semi-Fourier sequence. Thus they do not have the Budan-Fourier property either. This present paper deals with larger family of expressions and the results have the Budan-Fourier property.

In [3], Kovanskii studied zeros of Pfaffian functions. It involves sequence of functions defined by certain differential equations. The sequence is not semi-Fourier in general. As the result, it does not have the Budan-Fourier property. The present paper deals with smaller family of expressions. However, it generates semi-Fourier sequence with the Budan-Fourier property.

The paper is structured as follows: In Section 2, we give a precise statement of the problem that will be tackled in this paper. In Section 3, we provide an algorithm solving the problem. In Section 4, we prove the correctness of the algorithm. In Section 5, we provide several examples.

## 2 Problem

In this section, we state the main problem precisely. For this, we need several notions.

**Definition 1** (Exp-Int expression). An *exp-int expression* is an expression constructed with the variable symbol  $x$ , rational numbers, the binary operator symbols  $+$  and  $\cdot$  and the unary operator symbols  $\text{inv}$ ,  $\text{exp}$ , and  $\text{int}$ . The set of all exp-int expressions is denoted by  $EI$ .

Of course, the symbols  $\text{inv}$ ,  $\text{exp}$  and  $\text{int}$  stand for inverse, exponential and integration (anti-derivative) respectively. As usual, for convenience, we use the following usual short-hands:

$$f^n = \underbrace{f \cdot \dots \cdot f}_n \quad f^{-n} = \text{inv}(f)^n \quad f^0 = 1 \quad -e = -1 \cdot e$$

**Example 1.** We list several expressions that can be rewritten as Exp-Int expressions.

- $e^x \left( e^{\frac{1}{x} - e^{-x}} - e^{\frac{1}{x}} \right) + 5$   
 $\text{exp}(x)(\text{exp}(\text{inv}(x) - \text{exp}(-x)) - \text{exp}(\text{inv}(x))) + 5$
- $e^x \log x + e^{x^2} + x$   
 $\text{exp}(x) \text{int}(\text{inv}(x)) + \text{exp}(x^2) + x$

- $\frac{x}{\exp(x)-1} - \log(1 - \exp(-x)) - \frac{1}{x}$   
 $x \operatorname{inv}(\exp(x) - 1) - \operatorname{int}(\operatorname{inv}(\exp(x) - 1)) - \operatorname{inv}(x)$
- $2 \arccos\left(\frac{x}{2}\right) - \frac{\pi}{2} - \frac{1}{2}x\sqrt{4-x^2}$   
 $2 \operatorname{int}\left(-\exp\left(-\frac{1}{2} \operatorname{int}\left(-2x \operatorname{inv}(4-x^2)\right)\right)\right) - \frac{1}{2}x \exp\left(\frac{1}{2} \operatorname{int}\left(-2x \operatorname{inv}(4-x^2)\right)\right)$
- $\operatorname{Ei}(x^2+2) - \operatorname{li}(x^2+2) - e^{x^2+2}$   
 $\operatorname{int}\left(2x \exp(x^2+2) \operatorname{inv}(x^2+2)\right) - \operatorname{int}\left(2x \operatorname{inv}\left(\operatorname{int}\left(2x \operatorname{inv}(x^2+2)\right)\right)\right) - \exp(x^2+2)$

In the above, we have used the following translations. Note that integrals may include arbitrary constants.

$$\begin{aligned}
\log(x) &= \int \frac{1}{x} \\
\sqrt[n]{x} &= e^{\frac{1}{n} \log(x)} \\
\arccos(x) &= \int -\frac{1}{\sqrt{1-x^2}} \\
\operatorname{Ei}(x) &= \int \frac{e^x}{x} \\
\operatorname{li}(x) &= \int \frac{1}{\log(x)}
\end{aligned}$$

**Definition 2** (Expression differentiation). *Expression differentiation* is the operation  $D : EI \rightarrow EI$  defined recursively as follows.

1.  $D(x) = 1$ ,
2.  $D(c) = 0$ , for  $c \in \mathbb{Q}$ ,
3.  $D(f+g) = D(f) + D(g)$ ,
4.  $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$ ,
5.  $D(f^{-1}) = -f^{-2} \cdot D(f)$ ,
6.  $D(\exp(f)) = \exp(f) \cdot D(f)$ ,
7.  $D(\operatorname{int}(f)) = f$

*Remark 3.* We explicitly listed the above well known properties of differentiation, because we want to emphasize the obvious but crucial facts (that will be used later):

- The derivative (the result of the differentiation) of an exp-int expression is itself an exp-int expression.
- Let  $T \in \{\operatorname{inv}, \operatorname{int}, \exp\}$ . If  $T(g)$  is a subexpression of  $D(f)$ , then it is a subexpression of  $f$ .

*Remark 4.* An expression can be naturally viewed as a partial function, after fixing the meaning of the distinct int subexpressions by choosing anti-derivatives.

**Notation 5.** We write  $f \rightsquigarrow g$  iff for every choice of anti-derivatives corresponding to the distinct int subexpressions in  $f$  and  $g$  and for every  $x$  in the domain of  $f$ , we have  $f(x) = g(x)$ .

**Definition 6** (Semi-Fourier sequence). Let  $e \in EI$ . A sequence  $(g_1, h_1), \dots, (g_m, h_m)$  of pairs of elements of  $EI$  is a *semi-Fourier sequence* for  $e$  if

1.  $e \rightsquigarrow g_1 h_1^{-1}$
2.  $D(g_k) \rightsquigarrow g_{k+1} h_{k+1}^{-1}$ , for  $1 \leq k < m$ ,
3.  $D(g_m) \rightsquigarrow 0$ .

Now we are ready to state the problem precisely.

**Problem 1** (Main). *Devise an algorithm with the following specification.*

Input:  $e \in EI$

Output: a semi-Fourier sequence of  $e$

**Example 2.**

Input:  $e = \exp(x \text{ int}(\exp(-x^2))) - \text{int}(\exp(-x^2)) - 3$

Output:	$i$	$g_i$	$h_i$
	1	$f_3 - f_2 - 3$	1
	2	$(f_1^{-1} f_2 + x) f_3 - 1$	$f_1^{-1}$
	3	$f_2^2 + (2(x f_1) + 2x) f_2 + x^2 f_1^2 + 2f_1$	$f_3^{-1} f_1$
	4	$((-4x^2 + 4) f_1 + 2) f_2 + (-4x^3 + 4x) f_1^2 - 2(x f_1)$	1
	5	$(8x^3 - 16x) f_2 + (16x^4 - 32x^2 + 8) f_1 + 4x^2$	$f_1^{-1}$
	6	$(24x^2 - 16) f_2 + (-32x^5 + 136x^3 - 96x) f_1 + 8x$	1
	7	$48(x f_2) + (64x^6 - 432x^4 + 624x^2 - 112) f_1 + 8$	1
	8	$48f_2 + (-128x^7 + 1248x^5 - 2976x^3 + 1520x) f_1$	1
	9	$256x^8 - 3392x^6 + 12192x^4 - 11968x^2 + 1568$	$f_1^{-1}$
	10	$2048x^7 - 20352x^5 + 48768x^3 - 23936x$	1
	11	$14336x^6 - 101760x^4 + 146304x^2 - 23936$	1
	12	$86016x^5 - 407040x^3 + 292608x$	1
	13	$430080x^4 - 1221120x^2 + 292608$	1
	14	$1720320x^3 - 2442240x$	1
	15	$5160960x^2 - 2442240$	1
	16	$10321920x$	1
	17	$10321920$	1

where  $f_1 = \exp(-x^2)$   
 $f_2 = \text{int}(f_1)$   
 $f_3 = \exp(x f_2)$

### 3 Algorithm

In this section, we describe an algorithm that solves the problem posed in the previous section.

**Definition 7** (Equivalence). Let  $f, g \in EI$ . We say that  $f$  and  $g$  are *equivalent*, and write  $f \sim g$ , if  $f$  can be transformed into  $g$  by applying the commutative ring properties of  $+$  and  $\cdot$ . We write  $f \lesssim g$  if  $f$  is equivalent to a subexpression of  $g$ .

**Example 3.**

- $(x + 1)^2 \sim x^2 + 2x + 1$
- $\text{inv}(x) \lesssim x + \text{int}(\text{inv}(x))$

**Definition 8** (Extension tower). An *extension tower*  $E\langle f_1, \dots, f_r \rangle \subseteq EI$ , where  $r \geq 0$ , is defined recursively as follows

1.  $E\langle \rangle = \mathbb{Q}[x]$ ,
2. For  $1 \leq k \leq r$ ,  $f_k = T(p_k)$ ,  $p_k \in E\langle f_1, \dots, f_{k-1} \rangle$ , and

$$E\langle f_1, \dots, f_k \rangle = \begin{cases} \{a_n f_k^n + \dots + a_0 : a_0, \dots, a_n \in E\langle f_1, \dots, f_{k-1} \rangle\} & \text{if } T = \text{inv} \\ \{a_n f_k^n + \dots + a_0 : a_0, \dots, a_n \in E\langle f_1, \dots, f_{k-1} \rangle\} & \text{if } T = \text{int} \\ \{f_k^u(a_n f_k^n + \dots + a_0) : a_0, \dots, a_n \in E\langle f_1, \dots, f_{k-1} \rangle\} & \text{if } T = \text{exp} \end{cases}$$

where  $a_n \neq 0$  and if  $T = \text{exp}$  then  $a_0 \neq 0$  and  $u \in \mathbb{Z}$ .

Note that if  $f, g \in E\langle f_1, \dots, f_r \rangle$  then  $f + g$ ,  $fg$ , and  $D(f)$  can be easily transformed into equivalent elements of  $E\langle f_1, \dots, f_r \rangle$ . In the following when we write  $f + g$ ,  $fg$ , and  $D(f)$  for expressions  $f, g \in E\langle f_1, \dots, f_r \rangle$  we mean equivalent elements of  $E\langle f_1, \dots, f_r \rangle$ .

**Definition 9** (Rank). The *rank* of an exp-integrate expression is defined recursively as follows:

1.  $\text{rank}(x) = 0$ ,
2.  $\text{rank}(c) = 0$  for  $c \in \mathbb{Q}$ ,
3.  $\text{rank}(f + g) = \max(\text{rank}(f), \text{rank}(g))$ ,
4.  $\text{rank}(f \cdot g) = \max(\text{rank}(f), \text{rank}(g))$ ,
5.  $\text{rank}(\text{inv}(f)) = \text{rank}(f) + 1$ ,
6.  $\text{rank}(\text{exp}(f)) = \text{rank}(f) + 1$ ,
7.  $\text{rank}(\text{int}(f)) = \text{rank}(f) + 1$ .

**Algorithm 1 (EISF). — Main —**

Input:  $e \in EI$

Output: a semi-Fourier sequence of  $e$

1. Find a sequence  $e_1, \dots, e_r$  of all distinct subexpressions of  $e$  of the form  $T_i(u_i)$ , where  $T_i \in \{\text{inv}, \text{int}, \text{exp}\}$ , ordered by the non-decreasing value of rank.
2. Set  $f; f_1, \dots, f_r \leftarrow ET(e; e_1, \dots, e_r)$ .
3. Return  $ETSF(f; f_1, \dots, f_r)$ .

**Algorithm 2 (ET).**

Input:  $a_1, \dots, a_m; e_1, \dots, e_r$  such that

- $a_1, \dots, a_m \in EI$
- $e_1, \dots, e_r$  is a sequence containing all distinct subexpressions of  $a_1, \dots, a_m$  of the form  $T_i(u_i)$ , where  $T_i \in \{\text{inv}, \text{exp}, \text{int}\}$ , ordered by the increasing value of rank.

Output:  $b_1, \dots, b_m; f_1, \dots, f_r$  such that

- $f_i \sim e_i$  for  $1 \leq i \leq r$
- $b_j \sim a_j$  for  $1 \leq j \leq m$
- $b_1, \dots, b_m \in E\langle f_1, \dots, f_r \rangle$ .

1. If  $r = 0$ 
  - (a) For  $1 \leq j \leq m$  rewrite  $a_j$  as  $a_j \sim c_{j,n_j}x^{n_j} + \dots + c_{j,0} = b_j$ .
  - (b) For  $1 \leq j \leq m$  set  $b_j \leftarrow c_{j,n_j}x^{n_j} + \dots + c_{j,0}$ .
  - (c) Return  $(b_1, \dots, b_m; )$ .
2. If  $e_r = \text{inv}(v_r)$ 
  - (a) For  $1 \leq j \leq m$  rewrite  $a_j$  as  $a_j \sim c_{j,n_j}e_r^{n_j} + \dots + c_{j,0}$ , where  $c_{j,k}$  are free of  $e_r$ , for  $0 \leq k \leq n_j$ .
  - (b) Compute  $(d_{1,0}, \dots, d_{m,n_m}, w_r; f_1, \dots, f_{r-1}) = ET(c_{1,0}, \dots, c_{m,n_m}, v_r; e_1, \dots, e_{r-1})$ .
  - (c) Set  $f_r \leftarrow \text{inv}(w_r)$ .
  - (d) For  $1 \leq j \leq m$ , set  $b_j \leftarrow d_{j,n_j}f_r^{n_j} + \dots + d_{j,0}$ .
  - (e) Return  $(b_1, \dots, b_m; f_1, \dots, f_r)$ .
3. If  $e_r = \text{int}(v_r)$ 
  - (a) For  $1 \leq j \leq m$  rewrite  $a_j$  as  $a_j \sim c_{j,n_j}e_r^{n_j} + \dots + c_{j,0}$ , where  $c_{j,k}$  are free of  $e_r$ , for  $0 \leq k \leq n_j$ .
  - (b) Compute  $(d_{1,0}, \dots, d_{m,n_m}, w_r; f_1, \dots, f_{r-1}) = ET(c_{1,0}, \dots, c_{m,n_m}, v_r; e_1, \dots, e_{r-1})$ .
  - (c) Set  $f_r \leftarrow \text{int}(w_r)$ .
  - (d) For  $1 \leq j \leq m$ , set  $b_j \leftarrow d_{j,n_j}f_r^{n_j} + \dots + d_{j,0}$ .
  - (e) Return  $(b_1, \dots, b_m; f_1, \dots, f_r)$ .
4. If  $e_r = \text{exp}(v_r)$ 
  - (a) For  $1 \leq j \leq m$  rewrite  $a_j$  as  $a_j \sim e_r^{u_r}(c_{j,n_j}e_r^{n_j} + \dots + c_{j,0})$ , where  $c_{j,0} \neq 0$  and  $c_{j,k}$  are free of  $e_r$ , for  $0 \leq k \leq n_j$ .
  - (b) Compute  $(d_{1,0}, \dots, d_{m,n_m}, w_r; f_1, \dots, f_{r-1}) = ET(c_{1,0}, \dots, c_{m,n_m}, v_r; e_1, \dots, e_{r-1})$ .
  - (c) Set  $f_r \leftarrow \text{exp}(w_r)$ .
  - (d) For  $1 \leq j \leq m$ , set  $b_j \leftarrow d_r^{u_r}(d_{j,n_j}f_r^{n_j} + \dots + d_{j,0})$ .
  - (e) Return  $(b_1, \dots, b_m; f_1, \dots, f_r)$ .

**Algorithm 3** (*ETSF*).

**Input:**  $f; f_1, \dots, f_r$  such that  $f \in E\langle f_1, \dots, f_r \rangle$ .

**Output:** a semi-Fourier sequence  $(g_1, h_1), \dots, (g_m, h_m)$  of  $f$  such that  $g_i, h_i \in E\langle f_1, \dots, f_r \rangle$ , for  $1 \leq i \leq m$ .

1. If  $f$  does not contain any of  $f_1, \dots, f_r$ ,
  - (a) Set  $g_1 \leftarrow f = a_n x^n + \dots + a_0 \in \mathbb{Q}[x]$  and  $h_1 = 1$ .
  - (b) For  $1 \leq i \leq n$  set  $g_{i+1} \leftarrow D(g_i)$  and  $h_{i+1} \leftarrow 1$ .
  - (c) Return  $(g_1, h_1), \dots, (g_{n+1}, h_{n+1})$ .
2. Let  $k$  be maximal such that  $f_k \preceq f$ .
3. If  $f_k = \text{inv}(g)$ , let  $f = a_n f_k^n + \dots + a_0$ .
  - (a) Set  $t \leftarrow a_0 g^n + \dots + a_n$ .
  - (b) Compute  $(g_1, h_1), \dots, (g_m, h_m) \leftarrow \text{ETSF}(t; f_1, \dots, f_{k-1})$ .
  - (c) Return  $(g_1, g^n h_1), \dots, (g_m, h_m)$ .
4. If  $f_k = \text{int}(g)$ , let  $f = a_n f_k^n + \dots + a_0$ .
  - (a) Compute  $(e_1, h_1), \dots, (e_m, h_m) \leftarrow \text{ETSF}(a_n; f_1, \dots, f_{k-1})$ .
  - (b) For  $0 \leq i < n$ , set  $c_{i,1} \leftarrow a_i$ .
  - (c) For  $1 \leq j \leq m$ ,
    - i. Set  $g_j \leftarrow e_j f_k^n + c_{n-1,j} h_j f_k^{n-1} + \dots + c_{0,j} h_j$ .
    - ii. For  $0 \leq i < n-1$ , set  $c_{i,j+1} \leftarrow D(c_{i,j} h_j) + (i+1) c_{i+1,j} h_j g$ .
    - iii. Set  $c_{n-1,j+1} \leftarrow D(c_{n-1,j} h_j) + n e_j g$ .
  - (d) Set  $i \leftarrow n-1$ . While  $i \geq 0$  and  $c_{i,m+1} = 0$  decrement  $i$ .
  - (e) If  $i = -1$ , return  $(g_1, h_1), \dots, (g_m, h_m)$ .
  - (f) Set  $t \leftarrow c_{i,m+1} f_k^i + \dots + c_{0,m+1}$ .
  - (g) Compute  $(p_1, q_1), \dots, (p_l, q_l) \leftarrow \text{ETSF}(t; f_1, \dots, f_k)$ .
  - (h) Return  $(g_1, h_1), \dots, (g_m, h_m), (p_1, q_1), \dots, (p_l, q_l)$ .
5. If  $f_k = \text{exp}(g)$ , let  $f = f_k^u (a_n f_k^n + \dots + a_0)$ .
  - (a) Compute  $(e_1, h_1), \dots, (e_m, h_m) \leftarrow \text{ETSF}(a_0; f_1, \dots, f_{k-1})$ .
  - (b) For  $1 \leq i \leq n$ , set  $c_{i,1} \leftarrow a_i$ .
  - (c) For  $1 \leq j \leq m$ ,
    - i. Set  $g_j \leftarrow c_{n,j} h_j f_k^n + \dots + c_{1,j} h_j f_k + e_j$ .
    - ii. For  $1 \leq i \leq n$ , set  $c_{i,j+1} \leftarrow D(c_{i,j} h_j) + i c_{i,j} h_j D(g)$ .
  - (d) Set  $i \leftarrow 1$ . While  $i \leq n$  and  $c_{i,m+1} = 0$  increment  $i$ .
  - (e) If  $i = n+1$ , return  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m)$ .
  - (f) Set  $t \leftarrow c_{n,m+1} f_k^{n-i} + \dots + c_{i,m+1}$ .
  - (g) Compute  $(p_1, q_1), \dots, (p_l, q_l) \leftarrow \text{ETSF}(t; f_1, \dots, f_k)$ .
  - (h) Return  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m), (p_1, f_k^{-i} q_1), \dots, (p_l, q_l)$ .



**Example 4.** We illustrate the algorithm *EISF* by tracing it on the input  $e$  from Example 2.

$$e = \exp(x \operatorname{int}(\exp(-x^2))) - \operatorname{int}(\exp(-x^2)) - 3$$

1.  $e_1 = \exp(-x^2)$   
 $e_2 = \operatorname{int}(\exp(-x^2))$   
 $e_3 = \exp(x \operatorname{int}(\exp(-x^2)))$
2. Calling *ET* with  $(e; e_1, e_2, e_3)$ , we obtain

$$\begin{aligned} f &= f_3 - f_2 - 3 \\ f_1 &= \exp(-x^2) \\ f_2 &= \operatorname{int}(f_1) \\ f_3 &= \exp(x f_2) \end{aligned}$$

3. Calling *ETSF* with  $(f; f_1, f_2, f_3)$ , we obtain

$i$	$g_i$	$h_i$
1	$f_3 - f_2 - 3$	1
2	$(f_1^{-1} f_2 + x) f_3 - 1$	$f_1^{-1}$
3	$f_2^2 + (2(x f_1) + 2x) f_2 + x^2 f_1^2 + 2f_1$	$f_3^{-1} f_1$
4	$((-4x^2 + 4) f_1 + 2) f_2 + (-4x^3 + 4x) f_1^2 - 2(x f_1)$	1
5	$(8x^3 - 16x) f_2 + (16x^4 - 32x^2 + 8) f_1 + 4x^2$	$f_1^{-1}$
6	$(24x^2 - 16) f_2 + (-32x^5 + 136x^3 - 96x) f_1 + 8x$	1
7	$48(x f_2) + (64x^6 - 432x^4 + 624x^2 - 112) f_1 + 8$	1
8	$48 f_2 + (-128x^7 + 1248x^5 - 2976x^3 + 1520x) f_1$	1
9	$256x^8 - 3392x^6 + 12192x^4 - 11968x^2 + 1568$	$f_1^{-1}$
10	$2048x^7 - 20352x^5 + 48768x^3 - 23936x$	1
11	$14336x^6 - 101760x^4 + 146304x^2 - 23936$	1
12	$86016x^5 - 407040x^3 + 292608x$	1
13	$430080x^4 - 1221120x^2 + 292608$	1
14	$1720320x^3 - 2442240x$	1
15	$5160960x^2 - 2442240$	1
16	$10321920x$	1
17	$10321920$	1

**Example 5.** We illustrate the algorithm *ETSF* by tracing it on the input  $(f; f_1, f_2, f_3)$ , which is taken from Step 3 of Example 4.

$$\begin{aligned} f &= f_3 - f_2 - 3 \\ f_1 &= \exp(-x^2) \\ f_2 &= \operatorname{int}(f_1) \\ f_3 &= \exp(x f_2) \end{aligned}$$

The algorithm *ETSF* is recursive. Hence we trace the recursive calls.

$$\begin{array}{lcl}
\text{In}_1 & = & f_3 - f_2 - 3 \\
| \quad \text{In}_2 & = & -f_2 - 3 \\
| \quad | \quad \text{In}_3 & = & -f_1 \\
| \quad | \quad \text{Out}_3 & = & -1 \left| \begin{array}{c} f_1^{-1} \\ 1 \\ f_1^{-1} \end{array} \right. \\
| \quad \text{Out}_2 & = & -f_2 - 3 \left| \begin{array}{c} 1 \\ -1 \\ f_1^{-1} \end{array} \right. \\
| \quad \text{In}_4 & = & f_1^{-1} f_2^2 + ((2 + 2f_1^{-1})x) f_2 + x^2 f_1 + 2 \\
| \quad | \quad \text{In}_5 & = & f_1^{-1} \\
| \quad | \quad \text{Out}_5 & = & 1 \left| \begin{array}{c} f_1 \\ 1 \\ f_1^{-1} \end{array} \right. \\
| \quad | \quad \text{In}_6 & = & ((-4x^2 + 4) f_1 + 2) f_2 + (-4x^3 + 4x) f_1^2 - 2(x f_1) \\
| \quad | \quad | \quad \text{In}_7 & = & (-4x^2 + 4) f_1 + 2 \\
| \quad | \quad | \quad \text{Out}_7 & = & (-4x^2 + 4) f_1 + 2 \left| \begin{array}{c} 1 \\ 8x^3 - 16x \\ 24x^2 - 16 \\ 48x \\ 48 \end{array} \right| \begin{array}{c} f_1^{-1} \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \\
| \quad | \quad | \quad \text{In}_8 & = & (256x^8 - 3392x^6 + 12192x^4 - 11968x^2 + 1568) f_1 \\
| \quad | \quad | \quad \text{Out}_8 & = & 256x^8 - 3392x^6 + 12192x^4 - 11968x^2 + 1568 \left| \begin{array}{c} f_1^{-1} \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 10321920x \\ 10321920 \end{array} \right| \begin{array}{c} f_1^{-1} \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \\
| \quad | \quad \text{Out}_6 & = & ((-4x^2 + 4) f_1 + 2) f_2 + (-4x^3 + 4x) f_1^2 - 2(x f_1) \left| \begin{array}{c} 1 \\ f_1^{-1} \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right. \\
& & (8x^3 - 16x) f_2 + (16x^4 - 32x^2 + 8) f_1 + 4x^2 \\
& & (24x^2 - 16) f_2 + (-32x^5 + 136x^3 - 96x) f_1 + 8x \\
& & 48(x f_2) + (64x^6 - 432x^4 + 624x^2 - 112) f_1 + 8 \\
& & 48f_2 + (-128x^7 + 1248x^5 - 2976x^3 + 1520x) f_1 \\
& & \text{Out}_8 \\
| \quad \text{Out}_4 & = & f_2^2 + (2(x f_1) + 2x) f_2 + x^2 f_1^2 + 2f_1 \left| \begin{array}{c} f_1 \\ 1 \end{array} \right. \\
& & ((-4x^2 + 4) f_1 + 2) f_2 + (-4x^3 + 4x) f_1^2 - 2(x f_1) \\
& & \text{Out}_6 \\
\text{Out}_1 & = & f_3 - f_2 - 3 \left| \begin{array}{c} 1 \\ f_1^{-1} \\ f_3^{-1} f_1 \end{array} \right. \\
& & (f_1^{-1} f_2 + x) f_3 - 1 \\
& & f_2^2 + (2(x f_1) + 2x) f_2 + x^2 f_1^2 + 2f_1 \\
& & \text{Out}_6
\end{array}$$

## 4 Proof of Correctness of Algorithm

In this section, we prove that the main algorithm *EISF* is correct. We need to show the termination and the partial correctness. In the following subsections, we will show them one by one.

### 4.1 Termination

Since the main algorithm consists of a fixed sequence of steps, it suffices to show that the subalgorithms *ET* and *ETSF* terminate. The subalgorithm *ET* terminates since each recursion reduces the value of the natural number  $r$  until it becomes 0. Hence, it remains to show that the subalgorithm *ETSF* terminates. For this, it is convenient to introduce an ordering over  $E\langle f_1, \dots, f_r \rangle$ .

**Definition 10** (Ordering). Let  $d$  be the function from  $E\langle f_1, \dots, f_r \rangle$  to  $\mathbb{N}^2$  such that

$$d(h) = \begin{cases} (0, 0) & \text{if } h = 0, \\ (0, n) & \text{if } h = a_n x^n + \dots + a_0, \ a_n \neq 0, \\ (k, n) & \text{if } h = a_n f_k^n + \dots + a_0, \ a_n \neq 0, \ f_k = \text{inv}(p_k), \\ & \quad k \text{ is maximal such that } f_k \preceq f, \\ (k, n) & \text{if } h = a_n f_k^n + \dots + a_0, \ a_n \neq 0, \ f_k = \text{int}(p_k), \\ & \quad k \text{ is maximal such that } f_k \preceq f, \\ (k, n) & \text{if } h = f_k^u (a_n f_k^n + \dots + a_0), \ a_n \neq 0, \ a_0 \neq 0, \ u \in \mathbb{Z}, \\ & \quad f_k = \text{exp}(p_k), k \text{ is maximal such that } f_k \preceq f. \end{cases}$$

Let  $\prec$  be the binary relation over  $E\langle f_1, \dots, f_r \rangle$  such that

$$h_1 \prec h_2 \iff k_1 < k_2 \vee (k_1 = k_2 \wedge n_1 < n_2)$$

where  $(k_1, n_1) = d(h_1)$  and  $(k_2, n_2) = d(h_2)$ . □

It is obvious but crucial to note that  $E\langle f_1, \dots, f_r \rangle$  does not contain infinite  $\prec$ -descending chains. In the following we will show that when *ETSF* makes a recursive call, the first argument decreases with  $\prec$ . Let  $f \in E\langle f_1, \dots, f_r \rangle$  be such that it contains at least one of  $f_1, \dots, f_r$ . Let  $k$  be maximal such that  $f_k \preceq f$  and let  $d(f) = (k, n)$ . Note

(3b) *ETSF*( $t; f_1, \dots, f_{k-1}$ ).

Note that  $t = a_0 g^n + \dots + a_n$  where  $g, a_0, \dots, a_n \in E\langle f_1, \dots, f_{k-1} \rangle$ .

Hence  $d(t) = (k', n)$  where  $k' < k$ .

Thus  $d(t) \prec d(f)$ .

(4a) *ETSF*( $a_n; f_1, \dots, f_{k-1}$ ).

Note that  $a_n \in E\langle f_1, \dots, f_{k-1} \rangle$ .

Hence  $d(a_n) = (k', n')$  where  $k' < k$  and  $n'$  is a non-negative integer.

Thus  $d(a_n) \prec d(f)$ .

(4g) *ETSF*( $t; f_1, \dots, f_k$ ).

Note that  $t = c_{i,m+1} f_k^i + \dots + c_{0,m+1}$  where  $n-1 \geq i \geq 0$  and  $c_{i,m+1}, \dots, c_{0,m+1} \in E\langle f_1, \dots, f_{k-1} \rangle$ .

Hence if  $i > 0$  then  $d(t) = (k, i)$  else  $d(t) = d(c_{0,m+1}) = (k', n')$  where  $k' < k$  and  $n'$  is a non-negative integer.

Thus  $d(t) \prec d(f)$ .

(5a)  $ETSF(a_0; f_1, \dots, f_{k-1})$ .

Note that  $a_0 \in E\langle f_1, \dots, f_{k-1} \rangle$ .

Hence  $d(a_0) = (k', n')$  where  $k' < k$  and  $n'$  is a non-negative integer.

Thus  $d(a_0) \prec d(f)$ .

(5g)  $ETSF(t; f_1, \dots, f_k)$ .

Note that  $t = c_{n,m+1}f_k^{n-i} + \dots + c_{i,m+1}$  where  $n \geq i \geq 1$  and  $c_{n,m+1}, \dots, c_{i,m+1} \in E\langle f_1, \dots, f_{k-1} \rangle$ .

Hence if  $i < n$  then  $d(t) = (k, n-i)$  else  $d(t) = d(c_{n,m+1}) = (k', n')$  where  $k' < k$  and  $n'$  is a non-negative integer.

Thus  $d(t) \prec d(f)$ .

Hence the algorithm  $ETSF$  terminates. Thus the main algorithm  $EISF$  terminates.

## 4.2 Partial Correctness

In this subsection, we prove the partial correctness of the main algorithm  $EISF$ . Let  $e \in EI$ . We need to show that the output of the main algorithm  $EISF$  is a semi-Fourier sequence of  $e$ . The algorithm makes calls to the subalgorithms  $ET$  and  $ETSF$ . Hence we need to show their partial correctness. The partial correctness of the subalgorithm  $ET$  is obvious. Thus, we need to show the partial correctness of the subalgorithm  $ETSF$ .

**Definition 11** (Extension). We say that  $g$  extends  $f$ , and write  $f \rightsquigarrow g$ , if  $f$  be transformed into  $g$  by applying the commutative ring properties of  $+$  and  $\cdot$  and by cancellation (replacing  $e \cdot \text{inv}(e)$  with 1).

**Example 6.**  $(x^2 - 1)\text{inv}(x - 1) \rightsquigarrow x + 1$

For the purpose of the proof let us make the following definition.

**Definition 12** (SF sequence). Let  $f \in EI$ . A sequence  $(g_1, h_1), \dots, (g_m, h_m)$  of pairs of elements of  $EI$  is an *SF sequence* for  $f$  if

1.  $f h_1 \rightsquigarrow g_1$
2.  $D(g_k) h_{k+1} \rightsquigarrow g_{k+1}$ , for  $1 \leq k < m$ ,
3.  $D(g_m) \rightsquigarrow 0$ ,
4.  $h_k = t_{k,1} \cdots t_{k,l_k}$ , for  $1 \leq k \leq m$ , where for each  $1 \leq j \leq l_k$  one of the following two conditions holds
  - (a)  $t_{i,j}^{-1} \lesssim f$ ,
  - (b)  $t_{i,j} = \exp(u)^{-1}$  and  $\exp(u) \lesssim e$ .

It is easy to see that an SF sequence for  $f$  is a semi-Fourier sequence for  $f$ .

Let a sequence  $e_1, \dots, e_r$  of all distinct subexpressions of  $e$  of the form  $T_i(u_i)$ , where  $T_i \in \{\text{inv}, \exp, \text{int}\}$ , ordered by the non-decreasing value of *rank*. Let

$$f; f_1, \dots, f_r = ET(e; e_1, \dots, e_r)$$

Then we have

- $f_i \sim e_i$  for  $1 \leq i \leq r$
- $f \sim e$
- $f \in E\langle f_1, \dots, f_r \rangle$ .

We will show that the output of the subalgorithm *ETSF* is an SF sequence for  $f$ , and therefore a semi-Fourier sequence for  $f$ .

(1c) Return  $(g_1, h_1), \dots, (g_{n+1}, h_{n+1})$ .

Note that  $f \in \mathbb{Q}[x]$ . Thus the output is obviously an SF sequence of  $f$  (in fact the Fourier sequence of  $f$ ).

(3c) Return  $(g_1, g^n h_1), \dots, (g_m, h_m)$ .

Note  $f_k = \text{inv}(g)$  and  $t = a_n + \dots + a_0 g^n$  then  $d(t) = (k_1, n_1)$  with  $k_1 < k$ . By inductive hypothesis,  $(g_1, h_1), \dots, (g_m, h_m) = \text{ETSF}(t)$  is an SF sequence for  $t$ . Since

$$f g^n h_1 \rightsquigarrow t h_1 \rightsquigarrow g_1$$

and  $g^{-1} \lesssim f$ ,  $(g_1, g^n h_1), \dots, (g_m, h_m)$  is an SF sequence for  $f$ .

(4e) If  $i = -1$ , return  $(g_1, h_1), \dots, (g_m, h_m)$ .

Note  $f_k = \text{int}(g)$  and  $d(a_n) = (k_1, n_1)$  with  $k_1 < k$ . By inductive hypothesis,  $(e_1, h_1), \dots, (e_m, h_m) = \text{ETSF}(a_n)$  is an SF sequence for  $a_n$ . We have

$$f h_1 \sim a_n h_1 f_k^n + a_{n-1} h_1 f_k^{n-1} + \dots + a_0 h_1 \rightsquigarrow e_1 f_k^n + a_{n-1} h_1 f_k^{n-1} + \dots + a_0 h_1 = g_1$$

and, for  $1 \leq j \leq m$ ,

$$\begin{aligned} & D(g_j) \\ \sim & D(e_j f_k^n + c_{n-1,j} h_j f_k^{n-1} + \dots + c_{0,j} h_j) \\ \sim & D(e_j) f_k^n + (D(c_{n-1,j} h_j) + e_j n g) f_k^{n-1} + \\ & (D(c_{n-2,j} h_j) + c_{n-1,j} h_j (n-1) g) f_k^{n-2} + \dots + (D(c_{0,j} h_j) + c_{1,j} h_j g) \\ \sim & D(e_j) f_k^n + c_{n-1,j+1} f_k^{n-1} + c_{n-2,j+1} f_k^{n-2} + \dots + c_{0,j+1} \end{aligned}$$

Therefore, for  $1 \leq j < m$ ,

$$\begin{aligned} D(g_j) h_{j+1} & \sim D(e_j) h_{j+1} f_k^n + c_{n-1,j+1} h_{j+1} f_k^{n-1} + \dots + c_{0,j+1} h_{j+1} \\ & \rightsquigarrow e_{j+1} f_k^n + c_{n-1,j+1} h_{j+1} f_k^{n-1} + \dots + c_{0,j+1} h_{j+1} \\ & = g_{j+1} \end{aligned}$$

Moreover, since  $D(e_m) \rightsquigarrow 0$ ,

$$D(g_m) \rightsquigarrow c_{n-1,m+1} f_k^{n-1} + \dots + c_{0,m+1}$$

If the answer is returned in step (4e) then  $D(g_m) \rightsquigarrow 0$ , and hence  $(g_1, h_1), \dots, (g_m, h_m)$  is an SF sequence for  $f$ .

(4h) Return  $(g_1, h_1), \dots, (g_m, h_m), (p_1, q_1), \dots, (p_l, q_l)$ .

Otherwise

$$D(g_m) \rightsquigarrow c_{i,m+1} f_k^i + \dots + c_{0,m+1}$$

and, by inductive hypothesis,  $(p_1, q_1), \dots, (p_l, q_l)$  is an SF sequence for

$$t = c_{i,m+1} f_k^i + \dots + c_{0,m+1}$$

Hence

$$D(g_m) q_1 \rightsquigarrow t q_1 \rightsquigarrow p_1$$

and therefore  $(g_1, h_1), \dots, (g_m, h_m), (p_1, q_1), \dots, (p_l, q_l)$  is an SF sequence for  $f$ .

(5e) If  $i = n + 1$ , return  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m)$ .

Note  $f_k = \exp(g)$  and  $d(a_0) = (k_1, n_1)$  with  $k_1 < k$ . By inductive hypothesis,  $(e_1, h_1), \dots, (e_m, h_m) = ETSF(a_0)$  is an SF sequence for  $a_0$ . We have

$$f f_k^{-u} h_1 \rightsquigarrow a_n h_1 f_k^n + \dots + a_1 h_1 f_k + a_0 h_1 \rightsquigarrow a_n h_1 f_k^n + \dots + a_1 h_1 f_k + e_1 = g_1$$

and, for  $1 \leq j \leq m$ ,

$$\begin{aligned} & D(g_j) \\ \sim & D(c_{n,j} h_j f_k^n) + \dots + D(c_{1,j} h_j f_k) + D(e_j) \\ \sim & (D(c_{n,j} h_j) + c_{n,j} h_j n D(g)) f_k^n + \dots + (D(c_{1,j} h_j) + c_{1,j} h_j D(g)) + D(e_j) \\ \sim & c_{n,j+1} f_k^n + \dots + c_{1,j+1} f_k + D(e_j) \end{aligned}$$

Therefore, for  $1 \leq j < m$ ,

$$\begin{aligned} D(g_j) h_{j+1} & \sim c_{n,j+1} h_{j+1} f_k^n + \dots + c_{1,j+1} h_j f_k + D(e_j) h_j \\ & \rightsquigarrow c_{n,j+1} h_{j+1} f_k^n + \dots + c_{1,j+1} h_j f_k + e_{j+1} \\ & = g_{j+1} \end{aligned}$$

Moreover, since  $D(e_m) \rightsquigarrow 0$ ,

$$D(g_m) \rightsquigarrow c_{n,m+1} f_k^n + \dots + c_{1,m+1} f_k$$

If  $i = n + 1$  then  $D(g_m) \rightsquigarrow 0$ . Since  $f_k = \exp(g) \lesssim f$ ,  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m)$  is an SF sequence for  $f$ .

(5h) Return  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m), (p_1, f_k^{-i} q_1), \dots, (p_l, q_l)$ .

Otherwise

$$D(g_m) \rightsquigarrow (c_{n,m+1} f_k^{n-i} + \dots + c_{i,m+1}) f_k^i$$

and, by inductive hypothesis,  $(p_1, q_1), \dots, (p_l, q_l)$  is an SF sequence for

$$t = c_{n,m+1} f_k^{n-i} + \dots + c_{i,m+1}$$

Hence

$$\begin{aligned} D(g_m) f_k^{-i} q_1 & \rightsquigarrow t f_k^i f_k^{-i} q_1 \\ & \rightsquigarrow t q_1 \\ & \rightsquigarrow p_1 \end{aligned}$$

and therefore  $(g_1, f_k^{-u} h_1), \dots, (g_m, h_m), (p_1, f_k^{-i} q_1), \dots, (p_l, q_l)$  is an SF sequence for  $f$ .

## 5 Examples

**Example 7** (exp [5]). Find a semi-Fourier sequence of the function

$$f = e^{e^{e^x}} e^{-e^{e^x - e^{-e^x}}} - 10^5$$

Input expression:

$$e = \exp(\exp(\exp(x))) \exp(-\exp(\exp(x - \exp(-\exp(x))))) - 100000$$

Output:

$i$	$g_i$	$h_i$
1	$f_5 f_7 - 100000$	1
2	$(-f_2 - f_1^{-1}) f_3^{-1} f_4 f_6 + 1$	$f_1^{-1} f_3^{-1} f_5^{-1} f_7^{-1}$
3	$(-(f_1 f_2^2) + \dots - f_1^{-1}) f_4 - f_1 f_2^2 + \dots + 1$	$f_3 f_4^{-1} f_6^{-1}$
4	$(-(f_1 f_2^2) + \dots - 3f_1^{-1} + 2) f_4 + (2f_1 - 1) f_2 - 2f_1 + 4$	$f_1^{-1} f_2^{-1}$
5	$(-(f_1 f_2^3) + \dots + 2f_1^{-1}) f_4 + (-2f_1 + 3) f_2 - 2$	$f_1^{-1}$
6	$(-(f_1 f_2^3) + \dots + 2f_1 + 19f_1^{-1} - 14) f_4 + 2f_1 - 5$	$f_1^{-1} f_2^{-1}$
7	$(-(f_1 f_2^4) + \dots - 14f_1^{-1} + 4) f_4 + 2$	$f_1^{-1}$
8	$-(f_1^2 f_2^5) + \dots + 4$	$f_4^{-1}$
9	$(5f_1^2 - 2f_1) f_2^4 + \dots + 12f_1^2 + \dots + 232$	$f_1^{-1} f_2^{-1}$
10	$(-20f_1^2 + \dots - 2) f_2^4 + \dots + 24f_1 - 116$	$f_1^{-1}$
11	$(80f_1^2 + \dots + 26) f_2^4 + \dots + 24$	$f_1^{-1}$
12	$(-320f_1^2 + \dots - 216) f_2^3 + \dots + 120f_1^2 + \dots + 3032$	$f_1^{-1} f_2^{-1}$
13	$(960f_1^2 + \dots + 1256) f_2^3 + \dots + 240f_1 - 1304$	$f_1^{-1}$
14	$(-2880f_1^2 + \dots - 6232) f_2^3 + \dots + 240$	$f_1^{-1}$
15	$(8640f_1^2 + \dots + 28008) f_2^2 + \dots + 1296f_1^2 + \dots + 38776$	$f_1^{-1} f_2^{-1}$
16	$(-17280f_1^2 + \dots - 89712) f_2^2 + \dots + 2592f_1 - 15168$	$f_1^{-1}$
17	$(34560f_1^2 + \dots + 264096) f_2^2 + \dots + 2592$	$f_1^{-1}$
18	$(-69120f_1^2 + \dots - 732096) f_2 + 12096f_1^2 + \dots + 409392$	$f_1^{-1} f_2^{-1}$
19	$(69120f_1^2 + \dots + 1209024) f_2 + 24192f_1 - 149472$	$f_1^{-1}$
20	$(-69120f_1^2 + \dots - 1824192) f_2 + 24192$	$f_1^{-1}$
21	$69120f_1^2 + \dots + 2577600$	$f_1^{-1} f_2^{-1}$
22	$138240f_1 - 891648$	$f_1^{-1}$
23	138240	$f_1^{-1}$

where

$$\begin{aligned}
 f_1 &= e^x \\
 f_2 &= e^{-f_1} \\
 f_3 &= e^{f_1} \\
 f_4 &= e^{x-f_2} \\
 f_5 &= e^{f_3} \\
 f_6 &= e^{f_4} \\
 f_7 &= e^{-f_6}
 \end{aligned}$$

**Example 8** (exp-inv [5]). Find a semi-Fourier sequence of the function

$$f = e^x \left( e^{\frac{1}{x} - e^{-x}} - e^{\frac{1}{x}} \right) + 5$$

Input expression:

$$e = \exp(x)(\exp(\text{inv}(x) - \exp(-x)) - \exp(\text{inv}(x))) + 5$$

Output:

$i$	$g_i$	$h_i$
1	$f_2 f_5 - f_2 f_4 + 5$	1
2	$(-(x^2 f_3^2) + x^2 f_1 + x^2) f_4^{-1} f_5 - x^2 + 1$	$f_2^{-1} f_3^{-2} f_4^{-1}$
3	$(2(x^2 f_3^3) + \dots + x^2 f_1^2 + \dots + 2x) f_4^{-1} f_5 - 2x$	1
4	$(-6(x^2 f_3^4) + \dots + x^2 f_1^3 + \dots + 2) f_4^{-1} f_5 - 2$	1
5	$x^7 f_1^3 + \dots - x^5$	$f_1^{-1} f_3^{-5} f_4 f_5^{-1}$
6	$(-3x^7 + 7x^6) f_1^3 + \dots - 5x^4$	1
7	$(9x^7 + \dots + 42x^5) f_1^3 + \dots - 20x^3$	1
8	$(-27x^7 + \dots + 210x^4) f_1^3 + \dots - 60x^2$	1
9	$(81x^7 + \dots + 840x^3) f_1^3 + \dots - 120x$	1
10	$(-243x^7 + \dots + 2520x^2) f_1^3 + \dots - 120$	1
11	$(729x^7 + \dots + 5040x) f_1^2 + \dots + 4x^7 + \dots - 15120$	$f_1^{-1}$
12	$(-1458x^7 + \dots + 5040) f_1^2 + \dots + 28x^6 + \dots + 88200$	1
13	$(2916x^7 + \dots - 110880) f_1^2 + \dots + 168x^5 + \dots - 272160$	1
14	$(-5832x^7 + \dots + 1285200) f_1^2 + \dots + 840x^4 + \dots + 491400$	1
15	$(11664x^7 + \dots - 10503360) f_1^2 + \dots + 3360x^3 + \dots - 539280$	1
16	$(-23328x^7 + \dots + 68216400) f_1^2 + \dots + 10080x^2 + \dots + 355320$	1
17	$(46656x^7 + \dots - 375732000) f_1^2 + \dots + 20160x - 129600$	1
18	$(-93312x^7 + \dots + 1827226800) f_1^2 + \dots + 20160$	1
19	$(186624x^7 + \dots - 8060371200) f_1 - 320x^7 + \dots + 1040048640$	$f_1^{-1}$
20	$(-186624x^7 + \dots + 24812928000) f_1 - 2240x^6 + \dots - 1165919760$	1
21	$(186624x^7 + \dots - 68012179200) f_1 - 13440x^5 + \dots + 994933440$	1
22	$(-186624x^7 + \dots + 168875965440) f_1 - 67200x^4 + \dots - 635079360$	1
23	$(186624x^7 + \dots - 385341707520) f_1 - 268800x^3 + \dots + 293706240$	1
24	$(-186624x^7 + \dots + 817742822400) f_1 - 806400x^2 + \dots - 92962560$	1
25	$(186624x^7 + \dots - 1630128648960) f_1 - 1612800x + 18017280$	1
26	$(-186624x^7 + \dots + 3078168468480) f_1 - 1612800$	1
27	$186624x^7 + \dots - 5544580204800$	$f_1^{-1}$
28	$1306368x^6 + \dots + 4038444184320$	1
29	$7838208x^5 + \dots - 2349935400960$	1
30	$39191040x^4 + \dots + 1065791623680$	1
31	$156764160x^3 + \dots - 362821939200$	1
32	$470292480x^2 + \dots + 87160872960$	1
33	$940584960x - 13168189440$	1
34	$940584960$	1

where



$$\begin{aligned}
f_1 &= e^{-x} \\
f_2 &= e^x \\
f_3 &= \frac{1}{x} \\
f_4 &= e^{f_3} \\
f_5 &= e^{f_3 - f_1}
\end{aligned}$$

**Example 9** (exp-log (toy)). Find a semi-Fourier sequence of the function

$$f = e^x \log x + e^{x^2} + x$$

Input expression:

$$\exp(x) \operatorname{int}(\operatorname{inv}(x)) + \exp(x^2) + x$$

Output:

$i$	$g_i$	$h_i$
1	$f_4 + f_1^{-1} f_2 + x f_1^{-1}$	$f_1^{-1}$
2	$(2x^2 - x) f_2 + f_1 - x^2 + x$	$x f_1$
3	$(4x^3 + \cdots - 1) f_2 + f_1 - 2x + 1$	1
4	$(8x^4 + \cdots + 4) f_2 + f_1 - 2$	1
5	$(16x^5 + \cdots - 6) f_1^{-1} f_2 + 1$	$f_1^{-1}$
6	$32x^6 + \cdots + 54$	$f_1 f_2^{-1}$
7	$192x^5 + \cdots - 108$	1
8	$960x^4 + \cdots + 672$	1
9	$3840x^3 + \cdots - 912$	1
10	$11520x^2 + \cdots + 5568$	1
11	$23040x - 3840$	1
12	23040	1

where

$$\begin{aligned}
f_1 &= e^x \\
f_2 &= e^{x^2} \\
f_3 &= \frac{1}{x} \\
f_4 &= \log x
\end{aligned}$$

**Example 10** (exp-log [5]). Find a semi-Fourier sequence of the function

$$f = \frac{x}{\exp(x) - 1} - \log(1 - \exp(-x)) - \frac{1}{x}$$

Input expression:

$$x \operatorname{inv}(\exp(x) - 1) - \operatorname{int}(\operatorname{inv}(\exp(x) - 1)) - \operatorname{inv}(x)$$

Output:

$i$	$g_i$	$h_i$
1	$-f_4 + xf_3 - f_2$	1
2	$f_1^2 + (-x^3 - 2)f_1 + 1$	$f_2^{-2}f_3^{-2}$
3	$2f_1 - x^3 - 3x^2 - 2$	$f_1^{-1}$
4	$2f_1 - 3x^2 - 6x$	1
5	$2f_1 - 6x - 6$	1
6	$2f_1 - 6$	1
7	2	$f_1^{-1}$

where

$$\begin{aligned} f_1 &= e^x \\ f_2 &= \frac{1}{x} \\ f_3 &= -\frac{1}{1-f_1} \\ f_4 &= \log(1 - e^{-x}) \end{aligned}$$

**Example 11** (exp-log-arccos [6]). Find a semi-Fourier sequence of the function

$$f = 2 \arccos\left(\frac{x}{2}\right) - \frac{\pi}{2} - \frac{1}{2}x\sqrt{4-x^2}$$

Input expression:

$$2 \int (-\exp(-\frac{1}{2} \int (-2x \operatorname{inv}(4-x^2)))) - \frac{1}{2}x \exp(\frac{1}{2} \int (-2x \operatorname{inv}(4-x^2)))$$

Output:

$i$	$g_i$	$h_i$
1	$2f_5 - \frac{1}{2}(xf_4)$	1
2	$(\frac{1}{2}(x^2f_1) - \frac{1}{2})f_3^{-1}f_4 - 2$	$f_3^{-1}$
3	$2x$	$f_1^{-1}f_3f_4^{-1}$
4	2	1

where

$$\begin{aligned} f_1 &= \frac{1}{4-x^2} \\ f_2 &= \log(4-x^2) \\ f_3 &= e^{-\frac{f_2}{2}} \\ f_4 &= e^{\frac{f_2}{2}} \\ f_5 &= \arccos\left(\frac{x}{2}\right) - \frac{\pi}{4} \end{aligned}$$

**Example 12** (exp-int). Find a semi-Fourier sequence of the function

$$f = \operatorname{Ei}(x^2 + 2) - \operatorname{li}(x^2 + 2) - e^{x^2+2}$$

Input expression:

$$\int (2x \exp(x^2 + 2) \operatorname{inv}(x^2 + 2)) - \exp(x^2 + 2) - \int (2x \operatorname{inv}(\int (2x \operatorname{inv}(x^2 + 2))))$$

Output:

$i$	$g_i$	$h_i$
1	$-f_6 + f_4 - f_1$	1
2	$(-2x^3 - 2x) f_3 + (-2x^3 - 4x) f_1^{-1}$	$(x^2 + 2) f_1^{-1} f_3$
3	$(-6x^2 - 2) f_3 + (-4x^4 - 4x^2) f_2 + (4x^4 + \dots - 4) f_1^{-1}$	1
4	$-12(x f_3) + (8x^5 + 8x^3) f_2^2 + \dots + (-8x^5 + \dots + 12x) f_1^{-1}$	1
5	$-12f_3 + (-32x^6 - 32x^4) f_2^3 + \dots + (16x^6 + \dots + 12) f_1^{-1}$	1
6	$(-24x^7 + \dots - 1440x) f_1 - 32x^{15} + \dots - 4480x^3$	$(x^8 + \dots + 16) f_1$
7	$(-48x^8 + \dots - 1440) f_1 - 480x^{14} + \dots - 13440x^2$	1
8	$(-96x^9 + \dots - 12480x) f_1 - 6720x^{13} + \dots - 26880x$	1
9	$(-192x^{10} + \dots - 12480) f_1 - 87360x^{12} + \dots - 26880$	1
10	$(-384x^{11} + \dots - 187200x) f_1 - 1048320x^{11} + \dots - 645120x$	1
11	$(-768x^{12} + \dots - 187200) f_1 - 11531520x^{10} + \dots - 645120$	1
12	$(-1536x^{13} + \dots - 3951360x) f_1 - 115315200x^9 + \dots - 322560x$	1
13	$(-3072x^{14} + \dots - 3951360) f_1 - 1037836800x^8 + \dots - 322560$	1
14	$(-6144x^{15} + \dots - 105477120x) f_1 - 8302694400x^7 + \dots + 766402560x$	1
15	$(-12288x^{16} + \dots - 105477120) f_1 - 58118860800x^6 + \dots + 766402560$	1
16	$(-24576x^{17} + \dots - 3353011200x) f_1 - 348713164800x^5 + \dots + 29698099200x$	1
17	$(-49152x^{18} + \dots - 3353011200) f_1 - 1743565824000x^4 + \dots + 29698099200$	1
18	$(-98304x^{19} + \dots - 122464742400x) f_1 - 6974263296000x^3 - 199264665600x$	1
19	$(-196608x^{20} + \dots - 122464742400) f_1 - 20922789888000x^2 - 199264665600$	1
20	$(-393216x^{21} + \dots - 5023130112000x) f_1 - 41845579776000x$	1
21	$(-786432x^{22} + \dots - 5023130112000) f_1 - 41845579776000$	1
22	$-1572864x^{23} + \dots - 227809328947200x$	$f_1^{-1}$
23	$-36175872x^{22} + \dots - 227809328947200$	1
$\vdots$	$\vdots$	$\vdots$
45	$-40661706455989579897896960000$	1

where

$$\begin{aligned}
f &= -f_6 + f_4 - f_1 \\
f_1 &= e^{x^2+2} \\
f_2 &= \frac{1}{x^2+2} \\
f_3 &= \log(x^2 + 2) \\
f_4 &= \text{Ei}(x^2 + 2) \\
f_5 &= \frac{1}{f_3} \\
f_6 &= \text{li}(x^2 + 2)
\end{aligned}$$

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